# 2-COCYCLES ON THE LIE ALGEBRAS OF GENERALIZED DIFFERENTIAL OPERATORS

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ABSTRACT. In a recent paper by Zhao and the author, the Lie algebras $\mathcal{A}[\mathcal{D}] = \mathcal{A}\otimes \mathbb{F}[\mathcal{D}]$ of Weyl type were defined and studied, where $\mathcal{A}$ is a commutative associative algebra with an identity element over a field $\mathbb{F}$ of any characteristic, and $\mathbb{F}[\mathcal{D}]$ is the polynomial algebra of a commutative derivation subalgebra $\mathcal{D}$ of $\mathcal{A}$. In the present paper, the 2-cocycles of a class of the above Lie algebras $\mathcal{A}[\mathcal{D}]$ (which are called the Lie algebras of generalized differential operators in the present paper), with $\mathbb{F}$ being a field of characteristic 0, are determined. Among all the 2-cocycles, there is a special one which seems interesting. Using this 2-cocycle, the central extension of the Lie algebra is defined.

## §1. Introduction

We start with a brief definition. For a Lie algebra $L$ over a field $\mathbb{F}$ of characteristic zero, a *2-cocycle* on $L$ is a $\mathbb{F}$-bilinear function $\psi : L \times L \to \mathbb{F}$ satisfying the following conditions:

$$\psi(v_1, v_2) = -\psi(v_2, v_1), \tag{1.1}$$

$$\psi([v_1, v_2], v_3) + \psi([v_2, v_3], v_1) + \psi([v_3, v_1], v_2) = 0, \tag{1.2}$$

for $v_1, v_2, v_3 \in L$. Denote by $C^2(L,\mathbb{F})$ the vector space of 2-cocycles on $L$. For any $\mathbb{F}$-linear function $f : L \to \mathbb{F}$, one can define a 2-cocycle $\psi_f$ as follows

$$\psi_f(v_1, v_2) = f([v_1, v_2]), \tag{1.3}$$

for $v_1, v_2 \in L$. Such a 2-cocycle is called a *2-coboundary* or a *trivial 2-cocycle* on $L$. Denote by $B^2(L,\mathbb{F})$ the vector space of 2-coboundaries on $L$. A 2-cocycle $\phi$ is said to be *equivalent to* a 2-cocycle $\psi$ if $\phi - \psi$ is trivial. The quotient space

$$\begin{aligned} H^2(L,\mathbb{F}) &= C^2(L,\mathbb{F})/B^2(L,\mathbb{F}) \\ &= \{\text{the equivalent classes of 2-cocycles}\}, \end{aligned} \tag{1.4}$$

is called the *2-cohomology group* of $L$.

The 2-cocycles on Lie algebras play important roles in the central extensions of Lie algebras. It is well known that all 1-dimensional central extensions of $L$ are determined by the 2-cohomology group of $L$. Central extensions are often used in the structure theory and the representation theory of Kac-Moody algebras [4,5]. Using central extension, we can construct many infinite dimensional Lie algebras, such as affine Lie algebras, infinite

dimensional Heisenberg algebras, and generalized Virasoro and super-Virasoro algebras, which have a profound mathematical and physical background [2-4,6,7,12,15-17]. We can describe the structures and some of the representations of these Lie algebras by using central extension method [5,8]. Since the cohomology groups are closely related to the structures of Lie algebras, the computation of cohomology groups seems to be important and interesting as well. Berman [1] and Su [13] gave some computation of the low dimensional cohomology groups for some infinite dimensional Lie algebras.

Below, we shall introduce the Lie algebras we are going to consider in this paper. For any positive integer $n$, an additive subgroup $G$ of $\mathbb{F}^n$ is called *nondegenerate* if $G$ contains an $\mathbb{F}$-basis of $\mathbb{F}^n$. Let $\ell_1, ..., \ell_4$ be four nonnegative integers such that $\ell = \sum_{i=1}^4 \ell_i > 0$. For convenience, we denote

$$\ell'_i = \sum_{p=1}^{i} \ell_p, \quad \ell''_i = \sum_{p=i}^{4} \ell_p, \quad i = 1, 2, 3, 4. \tag{1.5}$$

For any $m, n \in \mathbb{Z}$, we use the notation

$$\overline{m, n} = \{m, m+1, ..., n\} \text{ if } m \leq n. \tag{1.6}$$

Take an additive subgroup $\Gamma$ of $\mathbb{F}^\ell$ such that for any $\alpha \in \Gamma$,

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_\ell) = (0, ..., 0, \alpha_{\ell_1+1}, ..., \alpha_\ell), \tag{1.7}$$

and such that $\Gamma$ is nondegenerate as a subgroup of $\mathbb{F}^{\ell''_2}$. Set

$$\vec{J} = \mathbb{Z}_+^{\ell'_2} \times \mathbb{Z}^{\ell_3} \times \{0\}^{\ell_4} \subset \mathbb{Z}^\ell. \tag{1.8}$$

Elements in $\mathbb{Z}^\ell$ will be written as

$$\vec{i} = (i_1, i_2, ..., i_\ell), \text{ and } \vec{i} = (i_1, ..., i_{\ell'_3}, 0, ..., 0) \text{ if } \vec{i} \in \vec{J}. \tag{1.9}$$

For any $i \in \mathbb{Z}$, $p \in \overline{1, \ell}$, we denote

$$i_{[p]} = (0, ..., \overset{p}{i}, 0, ..., 0) \in \mathbb{Z}^\ell. \tag{1.10}$$

Let $\mathcal{A}$ be the semi-group algebra $\mathbb{F}[\Gamma \times \vec{J}]$ with basis $\{x^{\alpha,\vec{i}} \,|\, (\alpha, \vec{i}) \in \Gamma \times \vec{J}\}$ and the algebraic operation

$$x^{\alpha,\vec{i}} \cdot x^{\beta,\vec{j}} = x^{\alpha+\beta,\vec{i}+\vec{j}}, \tag{1.11}$$

for $(\alpha, \vec{i}), (\beta, \vec{j}) \in \Gamma \times \vec{J}$. Denote the identity element $x^{0,\vec{0}}$ by 1, and for convenience, denote $x^\alpha = x^{\alpha,\vec{0}}$, $t^{\vec{i}} = x^{\vec{0},\vec{i}}$ for $\alpha \in \Gamma, \vec{i} \in \vec{J}$. Define the linear transformations $\{\partial_1^-, ..., \partial_{\ell'_3}^-, \partial_{\ell_1+1}^+, ..., \partial_\ell^+\}$ on $A$ by

$$\partial_p^-(x^{\alpha,\vec{i}}) = i_p x^{\alpha,\vec{i}-1_{[p]}}, \quad \partial_q^+(x^{\alpha,\vec{i}}) = \alpha_q x^{\alpha,\vec{i}}, \tag{1.12}$$

for all $p \in \overline{1, \ell_3'}$, $q \in \overline{\ell_1+1, \ell}$. The operators $\partial_p^-$ are called *down-grading operators* and $\partial_q^+$ are called *grading operators*. Set

$$\partial_p = \partial_p^-, \, \partial_q = \partial_q^- + \partial_q^+, \, \partial_r = \partial_r^+, \tag{1.13}$$

for $p \in \overline{1, \ell_1}$, $q \in \overline{\ell_1+1, \ell_3'}$, $r \in \overline{\ell_3'+1, \ell}$. Note that $\partial_p$ is locally nilpotent if $p \in \overline{1, \ell_1}$; locally finite if $p \in \overline{\ell_1+1, \ell_2'}$; not locally finite if $p \in \overline{\ell_2'+1, \ell_3'}$; semi-simple if $p \in \overline{\ell_3'+1, \ell}$.

Denote $\mathcal{D} = \sum_{p=1}^{\ell} \mathbb{F}\partial_p$. The $\mathbb{F}$-vector space

$$W = W(\ell_1, \ell_2, \ell_3, \ell_4, \Gamma) = \mathcal{A} \otimes \mathcal{D} = \sum_{p=1}^{\ell} \mathcal{A}\partial_p, \tag{1.14}$$

forms a Lie algebra under the bracket

$$[u\partial_p, v\partial_q] = u\partial_p(v)\partial_q - v\partial_q(u)\partial_p, \tag{1.15}$$

for $u, v \in A,\ p, q \in \overline{1, \ell}$, which is called a *Lie Algebra of generalized Witt type* [11,18,21].

Let $\mathbb{F}[\mathcal{D}]$ be the polynomial algebra with basis $\{\partial^\mu = \prod_{p=1}^{\ell} \partial_p^{\mu_p} \mid \mu \in \mathbb{Z}_+^\ell\}$ (where $\partial^\mu = 1$ if $\mu = 0$). The $\mathbb{F}$-vector space

$$\begin{aligned} \mathcal{W} &= \mathcal{W}(\ell_1, \ell_2, \ell_3, \ell_4, \Gamma) \\ &= \mathcal{A} \otimes \mathbb{F}[\mathcal{D}] = \operatorname{span}\{x^{\alpha,\vec{i}}\partial^\mu \mid \alpha \in \Gamma, \vec{i} \in \vec{J}, \mu \in \mathbb{Z}_+^\ell\}, \end{aligned} \tag{1.16}$$

forms an associative algebra under the algebraic operation

$$u\partial^\mu \odot v\partial^\nu = u \sum_{\lambda \in \mathbb{Z}_+^\ell} \binom{\mu}{\lambda} \partial^\lambda(v)\partial^{\mu+\nu-\lambda}, \tag{1.17}$$

where

$$\binom{\mu}{\lambda} = \prod_{p=1}^{\ell} \binom{\mu_p}{\lambda_p}, \quad \partial^\lambda(v) = \prod_{p=1}^{\ell} \partial_p^{\lambda_p}(v), \tag{1.18}$$

(cf. (1.12), (1.13)) for $u, v \in \mathcal{A}, \mu, \nu, \lambda \in \mathbb{Z}_+^\ell$, which induces a Lie algebra structure under the usual Lie bracket, called a *Lie Algebra of Weyl type* [19,20]. In this paper, we shall call $\mathcal{W}$ a *Lie Algebra of generalized differential operators* since it is a generalization of the Lie algebra of the differential operators considered in [4,9]. Note that using [19, Theorem 1.1], as in the proof of [20, Theorem 1.1], one can derive that $\mathcal{W}/\mathbb{F}$ is a simple Lie algebra, which is non-graded and non-linear in general.

It is well known that $W(0,0,0,1,\mathbb{Z})$ is the centerless Virasoro algebra and its 2-cohomology space is 1-dimensional such that the Virasoro algebra is its universal central extension [3]. The 2-cohomology space for general $W(\ell_1, \ell_2, \ell_3, \ell_4, \Gamma)$ was determined in [18]. The 2-cohomology space of the Lie algebra $\mathcal{W}(0,0,0,1,\mathbb{Z})$ of the differential operators, posed as an open problem by Kac [4], was determined by Li [9]. In [14], the author determined all the 2-cohomology spaces of $\mathcal{W}(0,0,0,n,\mathbb{Z}^n)$ for $n \geq 2$. In [10], Li and Wilson determined the 2-cohomology spaces of the Lie algebra of the derivations of

the formal Laurent polynomial ring $\mathbb{C}((t))$, the Lie algebra of the differential operators of $\mathbb{C}((t))$ and the Lie algebras of the differential operators of $\mathbb{C}((t))\otimes\mathbb{C}^n$.

In this paper, we shall determined all the 2-cohomology spaces of the Lie algebras $\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$. In [20], Zhao and the author determined the isomorphism classes of the Lie algebras $\mathcal{W}(0,\ell_2,0,\ell_4,\Gamma)$. However, for general $\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$, the determination of the isomorphism classes is still an unsolved problem. We hope that the computation of the 2-cohomology spaces of $\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$ may help to determine the isomorphism classes of $\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$. Our main result of the paper is stated in Theorem 3.1.

## §2. 2-Cocycles on $\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$

In this section, we shall consider 2-cocycles on the Lie algebra $\mathcal{W}=\mathcal{W}(\ell_1,\ell_2,\ell_3,\ell_4,\Gamma)$, i.e., we shall determine 2-cohomology group $H^2(\mathcal{W},\mathbb{F})$.

Take a basis of $\mathcal{W}$:

$$B=\{x^{\alpha,\vec{i}}\partial^{\mu}\mid(\alpha,\vec{i},\mu)\in\Gamma\times\vec{J}\times\mathbb{Z}_+^{\ell}\}. \tag{2.1}$$

We fix a $\tau\in\Gamma$ such that

$$\tau_p\neq 0, \tag{2.2}$$

for all $p\in\overline{\ell_1+1,\ell}$. For any $\mu=(\mu_1,...,\mu_\ell)\in\mathbb{Z}^{\ell}$, we define the *level* of $\mu$ to be $|\mu|=\sum_{p=1}^{\ell}\mu_p$. Define a total order on $\mathbb{Z}^{\ell}$ by: for $\mu,\nu\in\mathbb{Z}^{\ell}$, $\mu<\nu\Leftrightarrow$

$$|\mu|<|\nu| \text{ or } |\mu|=|\nu| \text{ and for the first } p \text{ with } \mu_p\neq\nu_p, \text{ we have } \mu_p<\nu_p. \tag{2.3}$$

**Lemma 2.1**. *Let $\psi$ be a 2-cocycle on $\mathcal{W}$. Then there exists a 2-cocycle $\phi$ equivalent to $\psi$ such that*

$$\phi(t^{1_{[p]}}\partial_p,x^{\alpha,\vec{i}}\partial^{\mu})=0 \quad \text{if } p\in\overline{1,\ell_3'}, \tag{2.4}$$

$$\phi(\partial_p,x^{\alpha,\vec{i}}\partial^{\mu})=0 \quad \text{if } p\in\overline{1,\ell}, \tag{2.5}$$

*for $(\alpha,\vec{i},\mu)\in\Gamma\times\vec{J}\times\mathbb{Z}_+^{\ell}$.*

*Proof.* Define a $\mathbb{F}$-linear function $f:\mathcal{W}\to\mathbb{F}$ as follows.

For $x^{\alpha,\vec{i}}\partial^{\mu}\in B$ with $\alpha\neq 0$, let $q$ be the minimal index such that $\alpha_q\neq 0$. Define $f(x^{\alpha,\vec{i}}\partial^{\mu})$ inductively on $|i_q|$ by

$$f(x^{\alpha,\vec{i}}\partial^{\mu})=\begin{cases}\alpha_q^{-1}(\psi(\partial_q,x^{\alpha,\vec{i}}\partial^{\mu})-i_qf(x^{\alpha,\vec{i}-1_{[q]}}\partial^{\mu})) & \text{if } i_q\geq 0,\\ -(1+\mu_q)^{-1}(\psi(t^{1_{[q]}}\partial_q,x^{\alpha,\vec{i}}\partial^{\mu})-\alpha_qf(x^{\alpha,\vec{i}+1_{[q]}}\partial^{\mu})) & \text{if } i_q=-1,\\ (i_q+1)^{-1}(\psi(\partial_q,x^{\alpha,\vec{i}+1_{[q]}}\partial^{\mu})-\alpha_qf(x^{\vec{\alpha},\vec{i}+1_{[q]}}\partial^{\mu})) & \text{if } i_q\leq -2.\end{cases} \tag{2.6}$$

Note that $[\partial_q,x^{\alpha,\vec{i}}\partial^{\mu}]=\alpha_qx^{\alpha,\vec{i}}\partial^{\mu}+i_qx^{\alpha,\vec{i}}\partial^{\mu}$ and $[t^1\partial_q,x^{\alpha,\vec{i}}\partial^{\mu}]=\alpha_qx^{\alpha,\vec{i}+1_{[q]}}\partial^{\mu}+(i_q-\mu_q)x^{\alpha,\vec{i}}\partial^{\mu}$. If $i_q\geq 1$, then $\vec{i}-1_{[q]}\in\vec{J}$, and so $x^{\vec{\alpha},\vec{i}-1_{[q]}}\partial_p\in B$; if $i_q\leq -1$, then $q\in\overline{\ell_2'+1,\ell_3'}$

(cf. (1.8)), and $t^{1_{[q]}}\partial_q \in \mathcal{W}$ and $x^{\alpha,\vec{i}+1_{[q]}}\partial^\mu \in B$, thus the right-hand side of (2.6) makes sense in all cases.

For $t^{\vec{i}}\partial^\mu \in B$ with $\vec{i} \neq 0$, let $r$ be the minimal index such that $i_r \neq 0$. Define

$$f(t^{\vec{i}}\partial^\mu) = \begin{cases} (i_r-\mu_r)^{-1}\psi(t^{1_{[r]}}\partial_r, t^{\vec{i}}\partial^\mu) & \text{if } i_r \neq \mu_r, \\ (i_r+1)^{-1}\psi(\partial_r, t^{\vec{i}+1_{[r]}}\partial^\mu) & \text{if } i_r = \mu_r \text{ (and so } i_r \geq 1). \end{cases} \tag{2.7}$$

Note that in (2.7), since $i_r \neq 0$, we have $r \leq \ell_3'$, and so all elements appearing in the right-hand side are in $\mathcal{W}$.

Finally, let $\partial^\mu \in B$ with $\mu \in \mathbb{Z}_+^\ell$. If $\ell_3' \neq 0$, we define

$$f(\partial^\mu) = \psi(\partial_1, t^{1_{[1]}}\partial^\mu). \tag{2.8}$$

On the other hand, if $\ell_3' = 0$, by using the formula

$$[x^\tau, x^{-\tau}\partial^{\mu+1_{[\ell]}}] = -(\mu_\ell+1)\partial^\mu - \sum_{0,1_{[\ell]}\neq\lambda\in\mathbb{Z}_+^\ell} \binom{\mu+1_{[\ell]}}{\lambda} \prod_{p=1}^{\ell} \tau_p^{\lambda_p}\partial^{\mu+1_{[\ell]}-\lambda}, \tag{2.9}$$

(cf. (1.17), (1.18)), we define $f(\partial^\mu)$ by induction on $\mu$ with respect to the order defined in (2.3):

$$\begin{aligned} f(\partial^\mu) = -(\mu_\ell+1)^{-1}\Big(&\psi(x^\tau, x^{-\tau}\partial^{\mu+1_{[\ell]}}) \\ &+ \sum_{0,1_{[\ell]}\neq\lambda\in\mathbb{Z}_+^\ell} \binom{\mu+1_{[\ell]}}{\lambda} \prod_{p=1}^{\ell} \tau_p^{\lambda_p} f(\partial^{\mu+1_{[\ell]}-\lambda})\Big), \end{aligned} \tag{2.10}$$

where, by ordering (2.3), we have $\mu + 1_{[\ell]} - \lambda < \mu$ for $\lambda \in \mathbb{Z}_+^\ell \backslash \{0, 1_{[\ell]}\}$.

Now set $\phi = \psi - \psi_f$ (cf. (1.3)). Then by (1.17), (2.6-8), (2.10), we have

$$\begin{aligned} &\phi(\partial_q, x^{\alpha,\vec{i}}\partial^\mu) = 0 && \text{if } \alpha \neq 0, && (2.11) \\ &\phi(t^{1_{[q]}}\partial_q, x^{\alpha,\vec{i}}\partial^\mu) = 0 && \text{if } \alpha \neq 0,\ i_q = -1, && (2.12) \\ &\phi(t^{1_{[r]}}\partial_r, t^{\vec{i}}\partial^\mu) = 0 && \text{if } i_r \neq \mu_r, && (2.13) \\ &\phi(\partial_r, t^{\vec{i}}\partial^\mu) = 0 && \text{if } i_r \geq 2, \mu_r = i_r - 1 \text{ or } r = 1, \vec{i} = 1_{[1]}, && (2.14) \\ &\phi(x^\tau, x^{-\tau}\partial^{\mu+1_{[\ell]}}) = 0 && \text{if } \ell_3' = 0, && (2.15) \end{aligned}$$

where

$$q = \min\{q \in \overline{\ell_1+1,\ell} \,|\, \alpha_q \neq 0\}, \quad r = \min\{r \in \overline{1,\ell_3'} \,|\, i_r \neq 0\}. \tag{2.16}$$

We prove (2.4), (2.5) in three cases.

*Case 1*: $\alpha \neq 0$.

Let $q$ be as in (2.16). By (2.11), (1.3), (1.17), we obtain

$$\begin{aligned} 0 &= \phi(\partial_q, \alpha_p x^{\alpha,\vec{i}+1_{[p]}}\partial^\mu + (i_p - \mu_p)x^{\alpha,\vec{i}}\partial^\mu) \\ &= \phi(\partial_q, [t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}}\partial^\mu]) \\ &= \delta_{q,p}\phi(\partial_q, x^{\alpha,\vec{i}}\partial^\mu) + \phi(t^{1_{[p]}}\partial_p, \alpha_q x^{\alpha,\vec{i}}\partial^\mu + i_q x^{\alpha,\vec{i}-1_{[q]}}\partial^\mu) \\ &= \alpha_q\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}}\partial^\mu) + i_q\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}-1_{[q]}}\partial^\mu), \end{aligned} \tag{2.17}$$

for $p \in \overline{1, \ell'_3}$, and

$$\begin{aligned} 0 &= \phi(\partial_q, \alpha_p x^{\alpha,\vec{i}}\partial^\mu + i_p x^{\alpha,\vec{i}-1_{[p]}}\partial^\mu) \\ &= \phi(\partial_q, [\partial_p, x^{\alpha,\vec{i}}\partial^\mu]) \\ &= \alpha_q\phi(\partial_p, x^{\alpha,\vec{i}}\partial^\mu) + i_q\phi(\partial_p, x^{\alpha,\vec{i}-1_{[q]}}\partial^\mu), \end{aligned} \tag{2.18}$$

for $p \in \overline{1, \ell}$. If $i_q \geq 0$, using (2.17), (2.18) and induction on $i_q$, we obtain (2.4), (2.5). So assume that $i_q < 0$. First suppose $i_q = -1$. Then $q \in \overline{\ell'_2 + 1, \ell'_3}$ (cf. (1.8)). If $p = q$, then by (2.11), (2.12), we have (2.4), (2.5). Assume that $p \neq q$. Using (2.12), we have

$$\begin{aligned} 0 &= \phi(t^{1_{[q]}}\partial_q, [t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}}\partial^\mu]) \\ &= \alpha_q\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}+1_{[q]}}\partial^\mu) - (1 + \mu_q)\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}}\partial^\mu) \\ &= -(1 + \mu_q)\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}}\partial^\mu), \end{aligned} \tag{2.19}$$

for $p \in \overline{1, \ell'_3}$, where the last equality follows from that $(\vec{i} + 1_{[q]})_q = i_q + 1 = 0$ and thus $\phi(t^{1_{[p]}}\partial_p, x^{\alpha,\vec{i}+1_{[q]}}\partial^\mu) = 0$ by the argument above. Thus (2.4) holds if $i_q = -1$. Then using (2.17) and induction on $|i_q|$, we see that (2.4) holds for all $i_q$. For (2.5), if $i_q = -1$, we have

$$\begin{aligned} 0 &= \phi(t^{1_{[q]}}\partial_q, [\partial_p, x^{\alpha,\vec{i}}\partial^\mu]) \\ &= -\delta_{p,q}\phi(\partial_p, x^{\alpha,\vec{i}}\partial^\mu) + \alpha_q\phi(\partial_p, x^{\alpha,\vec{i}+1_{[q]}}\partial^\mu) - (1 + \mu_q)\phi(\partial_p, x^{\alpha,\vec{i}}\partial^\mu) \\ &= -(1 + \delta_{p,q} + \mu_q)\phi(\partial_p, x^{\alpha,\vec{i}}\partial^\mu), \end{aligned} \tag{2.20}$$

for $p \in \overline{1, \ell}$, where the last equality follows again from that $(\vec{i} + 1_{[q]})_q = 0$. Thus (2.5) holds if $i_q = -1$. Then by (2.18) and induction on $|i_q|$, we deduce (2.5) for all $i_q$.

*Case 2*: $\alpha = 0, \vec{i} = 0$.

First, consider (2.5). If $p \in \overline{1, \ell'_3}$, then

$$0 = \phi(t^{1_{[p]}}\partial_p, [\partial_p, \partial^\mu]) = -(1 + \mu_p)\phi(\partial_p, \partial^\mu), \tag{2.21}$$

and we have (2.5) in this case. Thus assume that $p \in \overline{\ell'_3 + 1, \ell}$. If $\ell'_3 \geq 1$, then

$$\phi(\partial_p, \partial^\mu) = \phi(\partial_p, [\partial_1, t^{1_{[1]}}\partial^\mu]) = \phi(\partial_1, [\partial_p, t^{1_{[1]}}\partial^\mu]) = 0. \tag{2.22}$$

On the other hand, if $\ell_3'=0$, then by (2.15),

$$\begin{aligned}0 &= \tau_p\phi(x^\tau, x^{-\tau}\partial^{\mu+1_{[\ell]}})\\ &= \phi([\partial_p,x^\tau],x^{-\tau}\partial^{\mu+1_{[\ell]}}) = -\sum_{0\neq\lambda\in\mathbb{Z}_+^\ell}\binom{\mu+1_{[\ell]}}{\lambda}\prod_{q=1}^{\ell}\tau_q^{\lambda_q}\phi(\partial_p,\partial^{\mu+1_{[\ell]}-\lambda}),\end{aligned}\tag{2.23}$$

where the last equality follows from that $\phi([\partial_p,x^{-\tau}\partial^{\mu+1_{[\ell]}}],x^\tau)=0$ by (2.15). Using induction on $\mu$ in (2.23) gives $\phi(\partial_p,\partial^\mu)=0$. Next, consider (2.4). So suppose $p\in\overline{1,\ell_3'}$. Then $\ell_3'\geq 1$. If $p=1$, we have

$$\phi(t^{1_{[1]}}\partial_1,\partial^\mu)=2^{-1}\phi([\partial_1,t^{2_{[1]}}\partial_1],\partial^\mu)=2^{-1}\phi(\partial_1,[t^{2_{[1]}}\partial_1,\partial^\mu])=0,\tag{2.24}$$

where the last equality follows from (2.14), (2.22) and that

$$[t^{2_{[1]}}\partial_1,\partial^\mu]=-2\mu_1t^{1_{[1]}}\partial^\mu-\mu_1(\mu_1-1)\partial^{\mu-1_{[1]}}.\tag{2.25}$$

Thus assume that $p\in\overline{2,\ell_3'}$. If $\mu_1\neq 0$, then by (2.24),

$$0=\phi(t^{1_{[1]}}\partial_1,[t^{1_{[p]}}\partial_p,\partial^\mu])=\phi(t^{1_{[p]}}\partial_p,[t^{1_{[1]}}\partial_1,\partial^\mu])=-\mu_1\phi(t^{1_{[p]}}\partial_p,\partial^\mu).\tag{2.26}$$

On the other hand, if $\mu_1=0$, then

$$\begin{aligned}\phi(t^{1_{[p]}}\partial_p,\partial^\mu)&=\phi([\partial_1,t^{1_{[1]}+1_{[p]}}\partial_p],\partial^\mu)\\ &=\phi(\partial_1,[t^{1_{[1]}+1_{[p]}}\partial_p,\partial^\mu])=-\mu_p\phi(\partial_1,t^{1_{[1]}}\partial^\mu)=0,\end{aligned}\tag{2.27}$$

where the last equality follows from (2.14).

*Case 3*: $\alpha=0,\vec{i}\neq 0$.

Since $\vec{i}\neq 0$, we have $\ell_3'\geq 1$ (cf. (1.8)). Let $r$ be as in (2.16). First consider (2.4). If $i_r\neq\mu_r$, then by (2.13), we have

$$\begin{aligned}0 &= \phi(t^{1_{[r]}}\partial_r,[t^{1_{[p]}}\partial_p,t^{\vec{i}}\partial^\mu])\\ &= \phi(t^{1_{[p]}}\partial_p,[t^{1_{[r]}}\partial_r,t^{\vec{i}}\partial^\mu])=(i_r-\mu_r)\phi(t^{1_{[p]}}\partial_p,t^{\vec{i}}\partial^\mu),\end{aligned}\tag{2.28}$$

thus we have (2.4) in this case. So assume that $i_r=\mu_r$ (and so $i_r\geq 0$). Then

$$\begin{aligned}\phi(t^{1_{[p]}}\partial_p,t^{\vec{i}}\partial^\mu)&=(i_r+1)^{-1}\phi(t^{1_{[p]}}\partial_p,[\partial_r,t^{\vec{i}+1_{[r]}}\partial^\mu])\\ &=(i_r+1)^{-1}(-\delta_{r,p}+(i_p+\delta_{p,r}-\mu_p))\phi(\partial_r,t^{\vec{i}+1_{[r]}}\partial^\mu)\\ &=0,\end{aligned}\tag{2.29}$$

where the last equality follows from (2.13). This proves (2.4) in this case.

Next consider (2.5). We can write

$$t^{\vec{i}}\partial^\mu=\begin{cases}(i_1+1)^{-1}[\partial_1,t^{\vec{i}+1_{[1]}}\partial^\mu] & \text{if } i_1\neq -1,\\ -(1+\mu_1)^{-1}[t^{1_{[1]}}\partial_1,t^{\vec{i}}\partial^\mu] & \text{if } i_1=-1.\end{cases}\tag{2.30}$$

Thus by (1.2) and (2.4), we have

$$\phi(\partial_p, t^{\vec{i}}\partial^\mu) = \begin{cases} (i_1+1)^{-1}(i_p+\delta_{p,1})\phi(\partial_1, t^{\vec{i}+1_{[1]}-1_{[p]}}\partial^\mu) & \text{if } i_1 \neq -1, \\ -(1+\mu_1)^{-1}\delta_{p,1}\phi(\partial_1, t^{\vec{i}}\partial^\mu) & \text{if } i_1 = -1. \end{cases} \tag{2.31}$$

Thus it suffices to prove (2.5) for $p = 1$. Using (2.4), we have

$$0 = \phi(t^{1_{[1]}}\partial_1, [\partial_1, t^{\vec{i}}\partial^\mu]) = (-1 + i_1 - \mu_1)\phi(\partial_1, t^{\vec{i}}\partial^\mu). \tag{2.32}$$

By (2.32), it remains to consider the case $i_1 = \mu_1 + 1$. If $\mu_1 \geq 1$ or $\vec{i} = 1_{[1]}$, the result follows from (2.14). Thus assume $\mu_1 = 0$ and $\vec{i} \neq 1_{[1]}$. So $i_{r'} \neq 0$ for some $r' \in \overline{2, \ell'_3}$. Let $r' \in \overline{2, \ell'_3}$ be the minimal with $i_{r'} \neq 0$. If $i_{r'} \neq \mu_{r'}$, then by (2.4),

$$0 = \phi(t^{1_{[r']}}\partial_{r'}, [\partial_1, t^{\vec{i}}\partial^\mu]) = (i_{r'} - \mu_{r'})\phi(\partial_1, t^{\vec{i}}\partial^\mu). \tag{2.33}$$

On the other hand, if $i_{r'} = \mu_{r'}$ (and so $i_{r'} \geq 0$), then

$$\begin{aligned} \phi(\partial_1, t^{\vec{i}}\partial^\mu) &= (i_{r'}+1)^{-1}\phi(\partial_1, [\partial_{r'}, t^{\vec{i}+1_{[r']}}\partial^\mu]) \\ &= (i_{r'}+1)^{-1}\phi(\partial_{r'}, t^{\vec{i}-1_{[1]}+1_{[r']}}\partial^\mu) = 0, \end{aligned} \tag{2.34}$$

where the last equality follows from (2.14). This proves Lemma 2.1. ∎

Lemma 2.2. *If $\ell'_2 = \ell_1 + \ell_2 \geq 1$, then $H^2(\mathcal{W}, I\!\!F) = 0$.*

*Proof.* By Lemma 2.1, suppose $\phi$ is 2-cocycle on $\mathcal{W}$ satisfying (2.4), (2.5). We shall prove

$$\phi(x^{\alpha,\vec{i}}\partial^\mu, x^{\beta,\vec{j}}\partial^\nu) = 0, \tag{2.35}$$

for all $x^{\alpha,\vec{i}}\partial^\mu, x^{\beta,\vec{j}}\partial^\nu \in B$. By (2.5), we obtain

$$\begin{aligned} 0 &= \phi(\partial_1, [x^{\alpha,\vec{i}}\partial\mu, x^{\beta,\vec{j}}\partial^\nu]) \\ &= (\alpha_1+\beta_1)\phi(x^{\alpha,\vec{i}}\partial^\mu, x^{\beta,\vec{j}}\partial^\mu) \\ &\qquad + i_1\phi(x^{\alpha,\vec{i}-1_{[1]}}\partial^\mu, x^{\beta,\vec{j}}\partial^\nu) + j_1\phi(x^{\alpha,\vec{i}}\partial^\mu, x^{\beta,\vec{j}-1_{[1]}}\partial^\nu). \end{aligned} \tag{2.36}$$

Since $\ell'_2 \geq 1$, we have $i_1, j_1 \geq 0$ (cf. (1.8)). If $\alpha_1 + \beta_1 \neq 0$, then using induction on $i_1 + j_1$, we immediately get (2.35); otherwise, by substituting $\vec{i}$ by $\vec{i} + 1_{[1]}$ in (2.36), and using induction on $j_1 \geq 0$, we again deduce (2.35). ∎

## 3. The main results

Before given the main result of this paper, for convenience, we introduce the following notations

$$[\alpha]_\mu = \alpha(\alpha-1)\cdots(\alpha-\mu+1), \quad \binom{\alpha}{\mu} = (\mu!)^{-1}[\alpha]_\mu, \tag{3.1}$$

for $\alpha \in \mathbb{F}, \mu \in \mathbb{Z}_+$. We shall also use the following conventions

$$\alpha|_{\alpha=0} = \lim_{\alpha \to 0} \alpha, \tag{3.2}$$

$$\frac{1}{k!} = 0 \text{ if } k \leq -1, \tag{3.3}$$

where (3.2) shall be interpreted as follows: whenever we take a value $\alpha$ to be zero in an expression, we shall do it by taking the limit $\alpha \to 0$.

**Theorem 3.1**. *(1) If $\ell_1 + \ell_2 \neq 0$ or $\ell \geq 2$, then $H^2(\mathcal{W}, \mathbb{F}) = 0$.*

*(2) If $\ell = \ell_4 = 1$, then $H^2(\mathcal{W}, \mathbb{F}) = \mathbb{F}\overline{\phi}_0$, where $\overline{\phi}_0$ is the cohomology class of $\phi_0$ defined by*

$$\phi_0(x^\alpha \partial^\mu, x^\beta \partial^\nu) = \delta_{\alpha+\beta,0}(-1)^\mu \mu! \nu! \binom{\alpha+\mu}{\mu+\nu+1}, \tag{3.4}$$

*for all $\alpha, \beta \in \Gamma \subseteq \mathbb{F}$, $\mu, \nu \in \mathbb{Z}_+$, where $\partial = \partial_1$.*

*(3) Suppose $\ell = \ell_3 = 1$. Then for any $\gamma \in \Gamma$, there exists a cohomology class $\overline{\phi}_\gamma$ defined by*

$$\begin{aligned} &\phi_\gamma(x^{\alpha,i}\partial^\mu, x^{\beta,j}\partial^\nu) \\ &\quad = \delta_{\alpha+\beta,\gamma}(-1)^\mu \mu! \nu! \sum_{s=0}^{\mu+\nu+1} \binom{i}{s} \frac{\alpha^{\mu+\nu+1-s}}{(\mu+\nu+1-s)!} \cdot \frac{\gamma^{s-i-j-1}}{(s-i-j-1)!}, \end{aligned} \tag{3.5}$$

*for all $(\alpha, i, \mu), (\beta, j, \nu) \in \Gamma \times \mathbb{Z} \times \mathbb{Z}_+$ (if $\gamma = 0$, then by conventions (3.2), (3.3), $\frac{\gamma^{s-i-j-1}}{(s-i-j-1)!} = \delta_{i+j,s-1}$). Furthermore,*

$$H^2(\mathcal{W}, \mathbb{F}) = \prod_{\alpha \in \Gamma} \mathbb{F}\overline{\phi}_\gamma, \tag{3.6}$$

*is the direct product of all $\mathbb{F}\overline{\phi}_\gamma, \gamma \in \Gamma$.*

*Proof.* (1) follows from Lemma 2.2 if $\ell_2' \geq 1$. Next we suppose that $\ell_2' = 0$ and that $\phi$ is a 2-cocycle on $\mathcal{W}$ satisfying (2.4-5), (2.11-15). We shall divide the discussion into three cases.

*Case 1*: $\ell = \ell_4 = 1$.

Then $\mathcal{W} = \mathcal{W}(0,0,0,1,\Gamma) = \text{span}\{x^\alpha \partial^\mu \mid \alpha \in \Gamma, \mu \in \mathbb{Z}_+\}$ has the commutation relations (cf. (1.17))

$$[x^\alpha \partial^\mu, x^\beta \partial^\nu] = \sum_{s=0}^{\mu+\nu} \left(\binom{\mu}{s}\beta^s - \binom{\nu}{s}\alpha^s\right) x^{\alpha+\beta} \partial^{\mu+\nu-s}. \tag{3.7}$$

Note that when $\Gamma = \mathbb{Z}$, (3.4) is precisely the 2-cocycle defined in [4] and [9], thus as in [4], one can verify that, in general, (3.4) defines a nontrivial 2-cocycle on $\mathcal{W}(0,0,0,1,\Gamma)$.

If necessary by replacing $\Gamma$ by $\tau^{-1}\Gamma$, we can suppose that the element $\tau$ of $\Gamma$ defined in (2.2) is $\tau = 1$. Now let $\phi$ be a 2-cocycle satisfying (2.5), (2.15). Let $\phi_1 = \phi - \phi(x^1, x^{-1})\phi_0$. Then $\phi_1$ is a 2-cocycle satisfying (2.5), (2.15) and

$$\phi_1(x^1, x^{-1}) = 0. \tag{3.8}$$

We shall prove $\phi_1 = 0$. We have

$$(\alpha+\beta)\phi_1(x^\alpha\partial^\mu, x^\beta\partial^\nu) = \phi_1(\partial, [x^\alpha\partial^\mu, x^\beta\partial^\nu]) = 0, \tag{3.9}$$

$$\phi_1(x^\alpha, x^{-\alpha}) = \phi_1([x^{\alpha-1}\partial, x^1], x^{-\alpha}) = \alpha\phi_1(x^1, x^{-1}) = 0, \tag{3.10}$$

for $\alpha, \beta \in \Gamma, \mu, \nu \in \mathbb{Z}_+$. Thus $\phi_1(x^\alpha\partial^\mu, x^\beta\partial^\nu) = 0$ for $\alpha+\beta \neq 0$ or $\mu+\nu = 0$. Using induction on $\mu+\nu$, suppose that for $n \geq 0$ we have proved

$$\phi_1(x^\alpha\partial^\mu, x^\beta\partial^\nu) = 0, \tag{3.11}$$

for all $\alpha, \beta \in \Gamma$, $\mu, \nu \in \mathbb{Z}_+$ with $\mu+\nu \leq n$. Then by (3.7), (3.11),

$$\begin{aligned}\phi_1(x^\alpha\partial^{n+1}, x^{-\alpha}) &= (n+2)^{-1}\phi_1([x^{\alpha-1}\partial^{n+2}, x^1], x^{-\alpha})\\ &= \alpha\phi_1(x^1, x^{-1}\partial^{n+1})\\ &= 0,\end{aligned} \tag{3.12}$$

where the last equality follows from (2.15); and if $1 \leq \mu \leq n$,

$$\begin{aligned}\phi_1(x^\alpha\partial^\mu, x^{-\alpha}\partial^{n+1-\mu}) &= -((n+2-\mu)\alpha)^{-1}\phi_1(x^\alpha\partial^\mu, [\partial^{n+2-\mu}, x^{-\alpha}])\\ &= (n+2-\mu)^{-1}\mu\phi_1(\partial^{n+2-\mu}, \partial^{\mu-1})\\ &= 0,\end{aligned} \tag{3.13}$$

where the second equality follows from (3.12) and the last equality is obtained by induction on $\mu$. This proves that (3.11) holds for $\mu+\nu = n+1$. Hence $\phi_1 = 0$.

*Case 2*: $\ell = \ell_3 = 1$.

Then $\vec{J} = \mathbb{Z}$ and $\vec{i} = i$, and $\mathcal{W} = \mathcal{W}(0,0,1,0,\Gamma) = \mathrm{span}\{x^{\alpha,i}\partial^\mu \mid (\alpha, i, \mu) \in \Gamma \times \mathbb{Z} \times \mathbb{Z}_+\}$ has the commutation relations (cf. (1.17), (3.7))

$$\begin{aligned}[x^{\alpha,i}\partial^\mu, x^{\beta,j}\partial^\nu] &= \sum_{s=0}^{\mu+\nu}(\tbinom{\mu}{s}x^{\alpha,i}\partial^s(x^{\beta,j}) - \tbinom{\nu}{s}\partial^s(x^{\alpha,i})x^{\beta,j})\partial^{\mu+\nu-s}\\ &= \sum_{s=0}^{\mu+\nu}\sum_{r=0}^{s}(\tbinom{\mu}{s}\tbinom{s}{r}[j]_r\beta^{s-r} - \tbinom{\nu}{s}\tbinom{s}{r}[i]_r\alpha^{s-r})x^{\alpha+\beta,i+j-r}\partial^{\mu+\nu-s}.\end{aligned} \tag{3.14}$$

Let $\phi$ be a 2-cocycle satisfying (2.4), (2.5). Note that by (1.2), one can derive that $\phi(1, v) = 0$ for all $v \in \mathcal{W}$. Then by (2.4), (2.5), we have

$$\begin{aligned}0 &= \phi(\partial, [x^{\alpha,i}\partial^\mu, t^1]) = \phi([\partial, x^{\alpha,i}\partial^\mu], t^1]) + \phi(x^{\alpha,i}\partial^\mu, [\partial, t^1])\\ &= \phi(\partial(x^{\alpha,i})\partial^\mu, t^1]) = \alpha\phi(x^{\alpha,i}\partial^\mu, t^1) + i\phi(x^{\alpha,i-1}\partial^\mu, t^1),\end{aligned} \tag{3.15}$$

$$0 = \phi(t^1\partial, [x^{\alpha,i-1}\partial^\mu, t^1]) = \alpha\phi(x^{\alpha,i}\partial^\mu, t^1) + (i-\mu)\phi(x^{\alpha,i-1}\partial^\mu, t^1). \tag{3.16}$$

Comparing the coefficients of both terms of the right-hand sides of the above two equations, we deduce

$$\phi(x^{\alpha,i}\partial^\mu, t^1) = 0 \text{ if } \mu \neq 0, \quad \text{or } \alpha \neq 0, i \geq 0, \quad \text{or } \alpha = 0, i \neq -1, \tag{3.17}$$

$$\phi(x^{\alpha,i}, t^1) = -\frac{\alpha^{-i-1}}{(-i-1)!} b_\alpha \text{ if } \alpha \neq 0 \text{ and } i \leq -1, \tag{3.18}$$

where $b_\alpha = \phi(t^1, x^{\alpha,-1}) \in I\!\!F$ for $\alpha \in \Gamma$. Using convention (3.2), (3.3), we have

$$\phi(x^{\alpha,i}, t^1) = -\frac{\alpha^{-i-1}}{(-i-1)!} b_\alpha \text{ for all } (\alpha, i) \in \Gamma \times Z\!\!\!Z. \tag{3.19}$$

Then

$$\begin{aligned} \phi(x^{\alpha,i}, x^{\beta,j}\partial^\nu) &= \frac{1}{\nu+1}\phi(x^{\alpha,i}, [x^{\beta,j}\partial^{\nu+1}, t^1]) \\ &= -\frac{1}{\nu+1}\phi(\partial^{\nu+1}(x^{\alpha,i})x^{\beta,j}, t^1]) \\ &= \frac{1}{\nu+1}\sum_{s=0}^{\nu+1}\binom{\nu+1}{s}[i]_s\alpha^{\nu+1-s}\phi(t^1, x^{\alpha+\beta,i+j-s}) \\ &= \nu!\sum_{s=0}^{\nu+1}\binom{i}{s}\frac{\alpha^{\nu+1-s}}{(\nu+1-s)!}\cdot\frac{\gamma^{s-i-j-1}}{(s-i-j-1)!}b_\gamma, \end{aligned} \tag{3.20}$$

where $\gamma = \alpha+\beta$, and where, the second equality follows from (3.17) and the last equality follows from (3.19); and if $\mu > 0$,

$$\begin{aligned} \phi(x^{\alpha,i}\partial^\mu, x^{\beta,j}\partial^\nu) &= \frac{1}{\nu+1}\phi(x^{\alpha,i}\partial^\mu, [x^{\beta,j}\partial^{\nu+1}, t^1]) \\ &= \frac{\mu}{\nu+1}\phi(x^{\beta,j}\partial^{\nu+1}, x^{\alpha,i}\partial^{\mu-1}) \\ &= (-1)^\mu\frac{\mu!\nu!}{(\mu+\nu)!}\phi(x^{\alpha,i}, x^{\beta,j}\partial^{\mu+\nu}) \\ &= b_r \times \text{the right-hand side of (3.5)}, \end{aligned} \tag{3.21}$$

where the second equality follows from (3.17) and the third equality is obtained by induction on $\mu$ and the last equality follows from (3.20).

First assume that $\phi = \psi_f$ is a 2-coboundary defined by a linear function $f$ (cf. (1.3)). Then by (2.4), (2.5), we have

$$\alpha f(x^{\alpha,i}\partial^\mu) + (i-1-\mu)f(x^{\alpha,i-1}\partial^\mu) = \phi(t^1\partial, x^{\alpha,i-1}\partial^\mu) = 0, \tag{3.22}$$

$$\alpha f(x^{\alpha,i}\partial^\mu) + if(x^{\alpha,i-1}\partial^\mu) = \phi(\partial, x^{\alpha,i}\partial^\mu) = 0. \tag{3.23}$$

Thus $f = 0$, and so $\phi = 0$ and $b_\gamma = \phi(t^1, x^{\gamma,-1}) = 0$ for all $\gamma \in \Gamma$. Thus (3.20), (3.21) show that

$$\phi = \prod_{\gamma\in\Gamma} b_\gamma\phi_\gamma, \tag{3.24}$$

is the direct product of $b_\gamma\phi_\gamma$ for $\gamma\in\Gamma$, where $\phi_\gamma$ is defined in (3.5), i.e., we have (3.6) as long as each $\phi_\gamma$ is a 2-cocycle. It remains to prove that $\phi$ defined in (3.24) is a 2-cocycle for any $\beta_\gamma\in \mathbb{F},\gamma\in\Gamma$.

So now let $\varphi=\prod_{\gamma\in\Gamma} b_\gamma\phi_\gamma$. It is straightforward to check that formulas (3.15-21) hold for $\varphi$. By (3.21), the second equality of (3.20), we have

$$\begin{aligned}\varphi(x^{\alpha,i}\partial^\mu, x^{\beta,j}\partial^\nu) &= (-1)^\mu\frac{\mu!\nu!}{(\mu+\nu+1)!}\varphi(t^1,\partial^{\mu+\nu+1}(x^{\alpha,i})x^{\beta,j})\\ &= (-1)^{\nu+1}\frac{\mu!\nu!}{(\mu+\nu+1)!}\varphi(t^1,x^{\alpha,i}\partial^{\mu+\nu+1}(x^{\beta,j}))\\ &= -\varphi(x^{\beta,j}\partial^\nu, x^{\alpha,i}\partial^\mu),\end{aligned}\tag{3.25}$$

where the second equality follows from the third equality of (3.15) so that

$$\varphi(t^1,\partial(x^{\alpha,i})x^{\beta,j}) = -\varphi(t^1,x^{\alpha,i}\partial(x^{\beta,j})),\tag{3.26}$$

and the last equality of (3.25) follows from the first equality. This proves the skew-symmetry (1.1). To prove (1.2), recall that $\mathcal{W}$ is an associative algebra under algebraic operation (1.17). By (1.17) (cf. (3.14)), and the second equality of (3.25), we have

$$\begin{aligned}&\varphi(u\partial^\mu, v\partial^\nu\odot w\partial^\lambda)\\ &\quad=\sum_{s=0}^{\nu}(-1)^{\nu+\lambda+1-s}\frac{\mu!(\nu+\lambda-s)!}{(k+1-s)!}\binom{\nu}{s}\varphi(t^1,u\partial^{k+1-s}(v\partial^s(w))),\end{aligned}\tag{3.27}$$

for $u,v,w\in\mathcal{A}$, where $k=\mu+\nu+\lambda$. Using shifted version of (3.27) and (3.26), we have

$$\begin{aligned}&\varphi(v\partial^\nu, w\partial^\lambda\odot u\partial^\mu)\\ &\quad=\sum_{s=0}^{\lambda}(-1)^{\nu+s}\frac{\nu!(\lambda+\mu-s)!}{(k+1-s)!}\binom{\lambda}{s}\varphi(t^1,u\partial^s(\partial^{k+1-s}(v)w)),\end{aligned}\tag{3.28}$$

$$\begin{aligned}&\varphi(w\partial^\lambda, u\partial^\mu\odot v\partial^\nu)\\ &\quad=\sum_{s=0}^{\mu}(-1)^{\lambda}\frac{\lambda!(\nu+\mu-s)!}{(k+1-s)!}\binom{\mu}{s}\varphi(t^1,u\partial^s(v)\partial^{k+1-s}(w)).\end{aligned}\tag{3.29}$$

Denote the right-hand sides of (3.27), (3.28) and (3.29) by

$$\sum_{s=0}^{k+1}a_{p,s}\phi(t^1,u\partial^s(v)\partial^{k+1-s}(w))\ \text{for}\ p=1,2\ \text{and}\ 3\ \text{respectively.}$$

Using $(1+x)^{k+1-s}(1+x)^{-(\lambda+1)}=(1+x)^{\mu+\nu-s}$, we deduce the following binomial formula

$$\sum_q(-1)^q\binom{k+1-s}{\nu-q}\binom{\lambda+q}{q}=\binom{\mu+\nu-s}{\nu}.\tag{3.30}$$

Using (3.30), we can deduce that if $s \le \mu$, then

$$
\begin{aligned}
a_{1,s} &= \sum_{q=0}^{\nu}(-1)^{\nu+\lambda+1-q}\frac{\mu!(\nu+\lambda-q)!}{(k+1-q)!}\binom{\nu}{q}\binom{k+1-q}{s} \\
&= (-1)^{\lambda+1}\frac{\lambda!(\nu+\mu-s)!}{(k+1-s)!}\binom{\mu}{s} \\
&= -a_{3,s},
\end{aligned}
\tag{3.31}
$$

and $a_{2,s}=0$; and if $\mu < s \le \mu+\nu$, then $a_{1,s}=a_{2,s}=a_{3,s}=0$; and if $\mu+\nu < s \le k+1$, then

$$
\begin{aligned}
a_{1,s} &= \sum_{q=0}^{\nu}(-1)^{\nu+\lambda+1-q}\frac{\mu!(\nu+\lambda-q)!}{(k+1-q)!}\binom{\nu}{q}\binom{k+1-q}{s} \\
&= \sum_{q=0}^{\lambda}(-1)^{\nu+q+1}\frac{\nu!(\lambda+\mu-q)!}{(k+1-q)!}\binom{\lambda}{q}\binom{q}{k+1-s} \\
&= -a_{2,s},
\end{aligned}
\tag{3.32}
$$

and $a_{3,s}=0$. This proves that the sum of (3.27), (3.28) and (3.29) is zero. In particular, $\varphi$ satisfies (1.2). So, $\varphi$ is a 2-cocycle on $\mathcal{W}$.

*Case 3*: $\ell \ge 2$.

First suppose $\ell'_3 \ge 1$. As in (3.15-17), we have

$$\phi(t^{1_{[1]}}, x^{\alpha,\vec{i}}\partial^{\mu}) = 0 \tag{3.33}$$

if $\mu_1 \ne 0$. Assume that $\mu_1 = 0$. We can write

$$x^{\alpha,\vec{i}}\partial^{\mu} \equiv \frac{1}{\tau_\ell(\mu_\ell+1)}[x^{\alpha-\tau,\vec{i}}\partial^{\mu+1_{[\ell]}}, x^{\tau}] \ (\mathrm{mod}\, \mathcal{W}_{(\mu)}), \tag{3.34}$$

where $\mathcal{W}_{(\mu)} = \mathrm{span}\{x^{\alpha,\vec{i}}\partial^{\nu} \,|\, (\alpha,\vec{i},\nu) \in \Gamma \times \vec{J} \times \mathbb{Z}_+^{\ell}, \nu < \mu\}$ (cf. (2.3)). Then using induction on $\mu$, by (3.34), (1.2), we can prove that (3.33) also holds for $\mu_1 = 0$. Also as in (3.20), (3.21), we have

$$\phi(x^{\alpha,\vec{i}}\partial^{\mu}, x^{\beta,\vec{j}}\partial^{\nu}) = (\nu_1+1)^{-1}\phi([x^{\alpha,\vec{i}}\partial^{\mu}, x^{\beta,\vec{j}}\partial^{\nu+1_{[1]}}], t^{1_{[1]}}) = 0 \text{ if } \mu_1 = 0, \tag{3.35}$$

where the last equality follows from (3.33), and

$$\phi(x^{\alpha,\vec{i}}\partial^{\mu}, x^{\beta,\vec{j}}\partial^{\nu}) = -\frac{\mu_1}{\nu_1+1}\phi(x^{\alpha,\vec{i}}\partial^{\mu-1_{[1]}}, x^{\beta,\vec{j}}\partial^{\nu+1_{[1]}}) = 0 \text{ if } \mu_1 \ne 0. \tag{3.36}$$

where the last equality is obtained by induction on $\mu_1$. Thus $\phi = 0$.

Next suppose $\ell'_3 = 0$. By replacing $\partial_p$ by

$$\partial'_p = \partial_p - \frac{\tau_p}{\tau_\ell}\partial_\ell \text{ for } p = 1, ..., \ell-1, \tag{3.37}$$

we can suppose

$$\partial_p(x^{\tau}) = 0 \text{ for } p = 1, ..., \ell-1. \tag{3.38}$$

Then using (2.5), (2.15) and (1.2), we obtain

$$\phi(x^\alpha\partial^\mu, x^\beta\partial^\nu) = 0 \quad \text{if } \alpha+\beta\neq 0, \tag{3.39}$$

$$\phi(x^\tau, x^{-\tau}\partial^\mu) = 0 \quad \text{if } \mu_\ell \neq 0. \tag{3.40}$$

Assume that $\mu_\ell = 0$. Suppose we have proved that $\phi(x^\tau, x^{-\tau}\partial^\nu) = 0$ for $\nu < \mu$. Choose $\beta \in \Gamma$ with $\beta_{\ell-1} \neq 0$, then we have

$$\phi(x^\tau, x^{-\tau}\partial^\mu) = (\beta_{\ell-1}(\mu_{\ell-1}+1))^{-1}\phi(x^\tau, [x^{-\tau+\beta}\partial^{\mu+1_{[\ell-1]}}, x^\beta]) = 0, \tag{3.41}$$

where the first equality follows from the assumption that $\phi(x^\tau, x^{-\tau}\partial^\nu) = 0$ for $\nu < \mu$, and the last equality follows from (1.2) and (3.38). This proves that $\phi(x^\tau, x^{-\tau}\partial^\mu) = 0$ for $\mu \in \mathbb{Z}_+$. For any $\alpha \in \Gamma, \nu \in \mathbb{Z}_+^\ell$ with $\nu < \mu$, suppose we have proved that $\phi(x^\alpha, x^{-\alpha}\partial^\nu) = 0$. Then

$$\phi(x^\alpha, x^{-\alpha}\partial^\mu) = (\tau_\ell(\mu_\ell+1))^{-1}\phi(x^\alpha, [x^{-\alpha-\tau}\partial^{\mu+1_{[\ell]}}, x^\tau]) = 0, \tag{3.42}$$

where the first equality follows from the assumption that $\phi(x^\alpha, x^{-\alpha}\partial^\nu) = 0$ for $\nu < \mu$, and the last equality from (1.2), (3.40), (3.41) and the assumption. Finally,

$$\begin{aligned} \phi(x^\alpha\partial^\mu, x^{-\alpha}\partial^\nu) &= (\tau_\ell(\nu_\ell+1))^{-1}\phi(x^\alpha\partial^\mu, [x^{-\alpha-\tau}\partial^{\nu+1_{[\ell]}}, x^\tau]) \\ &= (\tau_\ell(\nu_\ell+1))^{-1}\phi(x^{-\alpha-\tau}\partial^{\nu+1_{[\ell]}}, [x^\alpha\partial^\mu, x^\tau]) \\ &= 0, \end{aligned} \tag{3.43}$$

where the first equality follows from induction on $\mu+\nu$, and the last equality follows from induction on $\mu$. Thus $\phi = 0$. ∎

We would like to conclude this paper by defining the central extension $\widehat{\mathcal{W}}(\Gamma)$ of the Lie algebra $\mathcal{W}(0,0,1,0,\Gamma)$. Let $\phi_0$ be the 2-cocycle defined by (3.5) with $\gamma = 0$. Then $\widehat{\mathcal{W}}(\Gamma)$ is the Lie algebra spanned by $\{L_{\alpha,i,\mu}, c \,|\, (\alpha,i,\mu) \in \Gamma\times\mathbb{Z}\times\mathbb{Z}_+\}$, with the following relations (cf. (3.14), (3.5))

$$\begin{aligned} &[L_{\alpha,i,\mu}, L_{\beta,j,\nu}] \\ &\quad= \sum_{s=0}^{\mu+\nu}\sum_{r=0}^{s}\left(\binom{\mu}{s}\binom{s}{r}[j]_r\beta^{s-r} - \binom{\nu}{s}\binom{s}{r}[i]_r\alpha^{s-r}\right)L_{\alpha+\beta,i+j-r,\mu+\nu-s} \\ &\qquad+\delta_{\alpha+\beta,0}(-1)^\mu\mu!\nu!\sum_{s=0}^{\mu+\nu+1}\binom{i}{s}\delta_{i+j,s-1}\frac{\alpha^{\mu+\nu+1-s}}{(\mu+\nu+1-s)!}c, \end{aligned} \tag{3.44}$$

and $[c, L_{\alpha,i,\mu}] = 0$, for $(\alpha,i,\mu), (\beta,j,\nu) \in \Gamma\times\mathbb{Z}\times\mathbb{Z}_+$.


## ACKNOWLEDGEMENT

The author was supported by a fund from Education Department of China.


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